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On the Residual Entropy of the Blume-Emery-Griffiths Model

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Abstract By means of a transfer matrix method, we show that the residual entropy S of the two dimensional square lattice Blume-Emery-Griffiths model on the antiquadrupolar-disordered and ferromagnetic-antiquadrupolar-disordered phase boundaries satisfies the inequalities $(\ln \lambda_{1,n})/(n+1) \leq S \leq (\ln \lambda_{1,n})/n$, where $\lambda_{1,n}$ is the largest eigenvalue of a transfer matrix F_n on a strip of width n. These bounds imply the existence of a $O(1/n)$ correction in the approximation of S by $(\ln \lambda_{1,n})/n$. Using these bounds, we calculate numerically the value of S, with precise estimates on the errors.

Keywords Residual entropy · Blume-Emery-Griffiths model · Transfer matrix

1 Introduction

The aim in this paper is to study the residual entropy of the two dimensional Blume-Emery-Griffiths model (BEG) [1]. The BEG Hamiltonian is formally given by

$$H(\sigma) = \sum_{\langle i,j \rangle} -J[\sigma_i \sigma_j + y\sigma_i^2 \sigma_j^2 + x(\sigma_i^2 + \sigma_j^2)] \equiv \sum_{\langle i,j \rangle} h(\sigma_i, \sigma_j), \tag{1}$$

where $i \in \mathbb{Z}^d$ is a lattice site, $\langle i, j \rangle$ is a nearest-neighbor bond, σ_i is the spin variable at site i, taking values 0, $+1$ and -1. The parameters J, x and y satisfy $J > 0$, $x, y \in \mathbb{R}$ and, without loss of generality, we will assume $J = 1$.

The residual entropy is defined by the limit

$$S = \lim_{|\Lambda| \to \infty} \frac{\ln N(\Lambda)}{|\Lambda|}, \tag{2}$$

where $N(\Lambda)$, which depends on x and y, is the number of spin configurations that minimizes the Hamiltonian (1) in a rectangular box $\Lambda \in \mathbb{Z}^d$, which we will denote by $H_\Lambda(\sigma)$, and

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$|\Lambda|$ is its volume. We remark that according to [2], see Corollary 3, the residual entropy (the entropy density at $T = 0$) is given by

$$\lim_{\Lambda\uparrow\infty} \sup_{b_\Lambda \in \Omega_{\Lambda^c}} \frac{1}{|\Lambda|} \ln N_\Lambda(b_\Lambda),$$

where $N_\Lambda(b_\Lambda)$ denotes the degeneracy of the ground state in Λ with boundary condition $b_\Lambda \in \Omega_{\Lambda^c} \equiv \{-1, 0, 1\}^{\Lambda^c}$. In our work, $N_\Lambda(b_\Lambda)$ is maximal for the zero (or free) boundary condition, which we denote by $N(\Lambda)$.

The residual entropy is the entropy which is present even after a substance is cooled to absolute zero. One of the first examples of residual entropy was pointed out by Linus Pauling [3] to describe water ice, which is an example of geometrically frustrated material. Geometrical frustration is an important feature in magnetism where it arises from the topographic arrangement of spins, forbidding the existence of a unique ground state. It is generated by the competition of different kinds of interaction and (or) by the lattice geometry. As a result, in the ground state not all bonds are not fully satisfied, and the state is highly degenerate. Well-known spin models describing such situations are, for instance, the Sherrington-Kirkpatrick model [4], the ANNNI model [5] and the Ising model on the square lattice with antiferromagnetic nearest-neighbor interaction in the maximum critical field [6].

The zero-temperature phase diagram of the BEG model in the xy-plane is obtained by minimizing the spin pair energy $h(\sigma_i, \sigma_j)$ and consists of three regions below (and their boundaries):

- $F = \{x, y \in \mathbb{R} : 1 + y + 2x > 0 \text{ and } 1 + y + x > 0\}$
- $D = \{x, y \in \mathbb{R} : 1 + y + 2x < 0 \text{ and } x < 0\}$
- $A = \{x, y \in \mathbb{R} : 1 + y + x < 0 \text{ and } x > 0\}$.

We consider those values of x and y for which $N(\Lambda)$ grows exponentially with the volume so that the limit (2) is non zero. We are particularly interested in the boundaries A–D ($x = 0$ and $y < -1$) and F–A–D ($x = 0$ and $y = -1$). We remark that the residual entropy in the A region is easy to compute and its value is $\ln 2/2$.

The problem of calculating the residual entropy is a non-trivial and old one. For instance, almost half a century ago, Brooks and Domb [6] estimated the entropy per spin for the antiferromagnetic Ising model in the maximum critical field $H_c = 4|J|$. Although the value they found has subsequently been established with higher precision [7–9], the exact solution for this problem has not been found so far. For the two dimensional BEG model in the region of parameters we will consider, as far as we know, the only known results are the ones obtained via Monte Carlo Method for the F–A–D phase boundary given in [10]. For the A–D phase boundary, there are not even numerical results. In Sect. 5 we give the numerical values of the residual entropy on these phase boundaries, with precise estimates on the errors.

In the study of antiferromagnetic Ising model in [9] the residual entropy is expressed as $S = \lim_{n\to\infty}(\ln \lambda_n)/n$, where λ_n is the largest eigenvalue of a matrix F_n, which plays the role of a transfer matrix. In this problem it was observed numerically that, in the approximation of S by $(\ln \lambda_n)/n$, there was a correction of order $1/n$. In our paper we use a transfer matrix method to show the existence of such a correction to the residual entropy for the two-dimensional BEG model on the A–D and F–A–D phase boundaries. This is the content of our Theorem 3.1. As an application of this theorem we give numerical values for the residual entropy on these phase boundaries, given in Sect. 5.

2 The Zero Temperature Transfer Matrix

The Hamiltonian of the BEG model is normalized so that $h(\sigma_i, \sigma_j) \geq 0$ for the values of x and y we will consider. A nearest neighbor spin pair (σ_i, σ_j) is a ground state pair if $h(\sigma_i, \sigma_j) = 0$ while a spin configuration σ is a ground state configuration if $H_\Lambda(\sigma) = 0$. Of course, $H_\Lambda(\sigma) = 0$ if and only if $h(\sigma_i, \sigma_j) = 0$ for all spin pairs. Then, one-dimensional ground states can be constructed by gluing together ground state pairs, two-dimensional ground states can be constructed by gluing together one-dimensional ground states and so forth. Here we fix $d = 2$ and consider $m \times n$ blocks $\Lambda_{m,n} \subset \mathbb{Z}^2$ of m rows and n columns. Let $G_1(1) \equiv \{-1, 0, 1\}$ and, for m or $n \geq 2$, let $G_m(n)$ be the set of all ground-state configurations of $H_{\Lambda_{m,n}}$ and $N_m(n) \equiv |G_m(n)|$, the cardinality of $G_m(n)$.

We say that the spin chains $\sigma, \sigma' \in G_1(n)$ are compatible if and only the pair (σ, σ'), seen as a spin configuration in $\Lambda_{2,n}$ with σ corresponding to one of its rows and σ' to the other one, is an element of $G_2(n)$. Let F_n be a $N_1(n) \times N_1(n)$ symmetric matrix defined as follows: $F_n(\sigma, \sigma') = 1$ if σ and σ' are compatible and $F_n(\sigma, \sigma') = 0$ otherwise. F_n will be the *transfer matrix* of our problem. Observe that the transfer matrix is related to $N_2(n)$ as follows:

$$N_2(n) = \sum_{\sigma,\sigma' \in G_1(n)} F_n(\sigma, \sigma').$$

More generally, for $m \geq 2$ the power $F_n^{m-1}(\sigma, \sigma')$ gives the number of ground-state configurations in $\Lambda_{m,n}$ that start with σ and end with σ' via the expression

$$N_m(n) = \sum_{\sigma,\sigma' \in G_1(n)} F_n^{m-1}(\sigma, \sigma'). \tag{3}$$

Fix a width n and consider the rectangle $\Lambda_{m,n}$ for large m. If we knew that the largest eigenvalue of F_n were positive and non-degenerate then the identity (3) relates the asymptotics of $N_m(n)$ for large m with the largest eigenvalue of F_n. This will be exploited in the next section within the context of the BEG model.

3 The $O(1/n)$ Correction at the A–D and F–A–D Phase Boundaries

In this section we show that the $O(1/n)$ correction, observed numerically in [9] for the antiferromagnetic Ising model with an external critical magnetic field $H_c = 4|J|$, also shows up for the BEG model at the A–D and F–A–D phase boundaries.

If we restrict ourselves to $x = 0$ and $y \leq -1$, we get the BEG model at the A–D and F–A–D phase boundaries. They correspond to $h(\sigma_i, \sigma_j) = -[\sigma_i\sigma_j + y\sigma_i^2\sigma_j^2]$ with $y < -1$ and $h(\sigma_i, \sigma_j) = -[\sigma_i\sigma_j - \sigma_i^2\sigma_j^2]$, respectively. Therefore, at the A–D phase boundary the ground state spin pairs will be $0+$, $0-$ and 00, while at the F–A–D phase boundary they will be $0+$, $0-$, 00, $++$ and $--$. The entries of the transfer matrices at the A–D and F–A–D phase boundaries will be

$$F_n(\sigma, \sigma') = \prod_{k=1}^{n} \delta_{\sigma_k\sigma'_k,0} \quad \text{and} \quad F_n(\sigma, \sigma') = \prod_{k=1}^{n} (\delta_{\sigma_k\sigma'_k,0} + \delta_{\sigma_k\sigma'_k,1}),$$

respectively, where $\delta_{i,j}$ is the usual Kronecker's delta. Motivated by the comments after (3), we now relate $N_1(n)$ to the largest eigenvalue of F_n. We enumerate the elements of $G_1(n)$

from 1 to $N_1(n)$ so that the ground state chain with all spins equal 0 will correspond to $i = 1$ (for an explicit enumeration of $G_1(n)$, see Sect. 4). It follows that $F_n(i, 1) = 1$ for all i, $F_n^2(i, j) \leq F_n^2(1, 1)$ for $i, j = 1, \ldots, N_1(n)$ and $F_n^2(1, 1) = N_1(n)$. Furthermore, since $F_n^2(i, j) = \sum_{k=1}^{N_1(n)} F_n(i, k) F_n(k, j) \geq F_n(i, 1) F_n(1, j) = 1$, we get that

$$1 \leq F_n^2(i, j) \leq N_1(n). \tag{4}$$

The lower bound in (4) implies that the largest eigenvalue of F_n^2 is positive and we denote it by $\mu_{1,n}$. This comes from the Perron-Frobenius theorem, see [11], as well as the bounds $\min_i \sum_{j=1}^{N_1(n)} F_n^2(i, j) \leq \mu_{1,n} \leq \max_i \sum_{j=1}^{N_1(n)} F_n^2(i, j)$, which, together with (4), imply

$$N_1(n) \leq \mu_{1,n} \leq N_1^2(n). \tag{5}$$

Actually, it follows also from a weaker version of the Perron-Frobenius that the largest eigenvalue of F_n is positive and we denote it by $\lambda_{1,n}$. Then $\mu_{1,n} = \lambda_{1,n}^2$ and from (5), we have

$$\sqrt{N_1(n)} \leq \lambda_{1,n} \leq N_1(n). \tag{6}$$

In Lemma 4.1 we will find $N_1(n)$ as a function of n. From there we conclude, together with (6), that $\lambda_{1,n}$ grows exponentially with n. The next theorem relates the residual entropy at A–D and F–A–D phase boundaries with $(\ln \lambda_{1,n})/n$:

Theorem 3.1 *The residual entropy S, defined by the limit (2), of the two-dimensional BEG model (1) at A–D and F–A–D phase boundaries satisfies the following bounds*:

$$\frac{\ln \lambda_{1,n}}{n+1} \leq S \leq \frac{\ln \lambda_{1,n}}{n}, \tag{7}$$

where $\lambda_{1,n}$ is the largest eigenvalue of the transfer matrix F_n. In particular, we get $S = \lim_{n\to\infty} (\ln \lambda_{1,n})/n$.

Proof We first show that the residual entropy on an infinite strip of width n, with any boundary conditions i, j, is given by

$$\frac{1}{n}\left[\lim_{m\to\infty} \frac{\ln F_n^m(i, j)}{(m+1)}\right] = \frac{\ln \lambda_{1,n}}{n}. \tag{8}$$

Since F_n^2 is a real symmetric matrix then $U_n^t F_n^2 U_n = D_n$, where $U_n = [V_1 \ \ldots \ V_{N_1(n)}]$ and D_n the diagonal matrix $D_n(k, j) = \delta_{k,j} \mu_{k,n}$, where $\mu_{k,n}$ is the eigenvalue associated to the orthonormalized eigenvectors $\{V_k\}$. It follows again from the Perron-Frobenius theorem when applied to F_n^2 that its largest positive eigenvalue, $\mu_{1,n}$, is non-degenerate, $\mu_{1,n} > |\mu_{k,n}|$ for all $k > 1$ and that the corresponding eigenvector, V_1, has positive entries, i.e., $U_n(k, 1) > 0$, for all k. In order to prove (8), we restrict to even m's. For odd m's, similar arguments also hold. Then, since $\mu_{1,n} = \lambda_{1,n}^2$:

$$F_n^{2m}(i, j) = (U_n D_n^m U_n^t)(i, j) = \lambda_{1,n}^{2m} E_{n,m}(i, j),$$

where

$$E_{n,m} = U_n(i, 1) U_n(j, 1) + \sum_{l\neq 1} U_n(i, l) \left(\frac{\mu_{l,n}}{\mu_{1,n}}\right)^m U_n(j, l).$$

Then (8) follows from the equation above because $|\mu_{l,n}/\mu_{1,n}| < 1$ for all $l \neq 1$.

To proceed, we take the infinite-volume limit through strips of width n as follows: for fixed n and for k, m any positive integers, consider only those ground-state configurations in $\Lambda_{2m+1,k(n+1)}$, having 0's in all their columns which are multiples of $n+1$. The number of such configurations is $[N_{2m+1}(n)]^k$. Therefore

$$N_{2m+1}(k(n+1)) > [N_{2m+1}(n)]^k \geq [F_n^{2m}(1,1)]^k,$$

which, together with (8), imply that

$$S = \lim_{m,k\to\infty}\left[\frac{\ln[N_{2m+1}(k(n+1))]}{(2m+1)k(n+1)}\right] \geq \frac{1}{n+1}\lim_{m\to\infty}\left[\frac{\ln F_n^{2m}(1,1)}{(2m+1)}\right] = \frac{\ln\lambda_{1,n}}{n+1}.$$

On the other hand, notice that

$$N_{2m+1}(kn) < [N_{2m+1}(n)]^k \leq [N_1(n)]^{2k}[(F_n^2)^m(1,1)]^k, \tag{9}$$

where the second inequality follows from

$$N_{m+1}(n) = \sum_{i,j=1}^{N_1(n)} F_n^m(i,j) = F_n^m(1,1)\sum_{i,j=1}^{N_1(n)} \frac{F_n^m(i,j)}{F_n^m(1,1)} \leq F_n^m(1,1)N_1^2(n),$$

since $F_n^m(i,j)/F_n^m(1,1) \leq 1$ for all i, j. Then, from (8) and (9), we have

$$S = \lim_{m,k\to\infty}\left[\frac{\ln N_{2m+1}(kn)}{(2m+1)kn}\right] \leq \frac{1}{n}\lim_{m\to\infty}\left[\frac{\ln F_n^{2m}(1,1)}{(2m+1)}\right] = \frac{\ln\lambda_{1,n}}{n}. \qquad \square$$

Remark The upper and lower bounds in (7) imply that $(\ln\lambda_{1,n})/n$ approaches S from above and that $(\ln\lambda_{1,n})/n$ is an approximation for S within an error which is bounded from above by $(\ln\lambda_{1,n}/n)/(n+1) \leq C/(n+1)$, where C is a constant smaller then $\ln 3$.

4 On the Matrix Structure of F_n

In Sect. 5 we use (7) to provide numerical values for S. To perform the numerical calculations, we need to represent F_n as a matrix in some suitable basis of a $N_1(n)$ dimensional vector space. Observing that the ground state spin chains in $G_1(n)$ are linearly independent vectors in this space, in this section we will prescribe an order in $G_1(n)$ so that a suitable representation for F_n can be handled. To do so we first find $N_1(n)$ as a function of n for the A–D and F–A–D phase boundaries. As a byproduct of the matrix structure of F_n, we will observe that F_n has a fractal, self-similar structure, a phenomena that has been also observed in [9] for the antiferromagnetic Ising model.

Lemma 4.1 *In the A–D and F–A–D phase boundaries, we have the following expressions for* $N_1(n)$, *respectively*:

$$N_1(n) = \frac{2^{n+2} + (-1)^{n+1}}{3}, \tag{10}$$

$$N_1(n) = \frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{2}. \tag{11}$$

Proof Let $\Lambda_{1,n} = \{x_1, \ldots, x_n\}$, with $x_{i+1} = x_i + 1$. The proof is based on the generating function method. For the A–D phase boundary we find a recursion relation satisfied by $N_1(n)$ and so its generating function. For the F–A–D phase boundary, we find a system of two equations which couples the recursion relations of $N_1(n)$ and $N_1(n, +)$, where for $\rho = 0, \pm 1$, we define $N_1(n, \rho)$ as the number of elements $\sigma \in G_1(n)$ such that $\sigma_{x_n} = \rho$. From this system of equations we find explicit expressions for the generating functions for $N_1(n)$ and $N_1(n, +)$.

Since $G_1(1) = \{0, +, -\}$, then $N_1(1) = 3$. Also, for $n \geq 2$, $N_1(n, 0) = N_1(n-1)$.

(i) For the A–D phase boundary, for $n \geq 3$, $N_1(n, -) = N_1(n, +) = N_1(n-1, 0) = N_1(n-2)$ which, together with $N_1(n) = N_1(n, 0) + N_1(n, +) + N_1(n, -)$ for any $n \geq 2$, implies that for any $n \geq 3$

$$N_1(n) = N_1(n-1) + 2N_1(n-2), \tag{12}$$

which also holds for $n \geq 2$ if we adopt the convention that $N_1(0) = 1$. By straightforward calculation, we find from (12) that the generating function for $N_1(n)$ is given by $f(x) = (1 + 2x)(1 - x - 2x^2)^{-1}$, which implies (10).

(ii) For the F–A–D phase boundary, notice that $N_1(2) = 7$, $N_1(2, +) = 2$ and, for $n \geq 3$, we have the following recursion relations:

$$N_1(n) = N_1(n, 0) + N_1(n, +) + N_1(n, -) = N(n-1) + 2N_1(n, +),$$

$$N_1(n, +) = N_1(n-1, 0) + N_1(n-1, +) = N_1(n-2) + N_1(n-1, +).$$

For $n = 0, 1, \ldots$, let $a_n = N_1(n+1, 0)$ and $b_n = N_1(n+1)$. Then $a_o = 1$, $a_1 = 2$, $b_o = 3$, $b_1 = 7$ and, for $n \geq 2$, the above recursion relations give

$$a_n = a_{n-1} + b_{n-2} \quad \text{and} \quad b_n = b_{n-1} + 2a_n.$$

Then the generating functions for a_n and b_n are $f(x) = -(x^2 + 2x - 1)^{-1}$ and $g(x) = -(x+3)(x^2 + 2x - 1)^{-1}$, respectively. From the expression for $g(x)$ we have

$$b_n = N_1(n+1) = \frac{(1+\sqrt{2})^{n+2} + (1-\sqrt{2})^{n+2}}{2},$$

which implies (11). □

As a consequence of Lemma 4.1, we conclude that on the A–D and F–A–D phase boundaries, the one-dimensional residual entropies are $\ln 2$ and $\ln(1+\sqrt{2})$, respectively.

We now order the set $G_1(n)$ so that a suitable representation of F_n can be handled. To make the notation simpler, we will denote $G_1(n)$ just by G_n. We fix attention to the A–D phase boundary but a similar procedure holds for the F–A–D phase boundary. We begin by considering the disjoint union $G_{n,0} \cup G_{n,+} \cup G_{n,-} = G_n$, where $G_{n,\sigma}$ consists of those elements of G_n such that $\sigma_{x_n} = \sigma$. Next, we will see the set G_n as a $N_1(n) \times N_1(n)$ matrix so that each of its rows corresponds to a unique element of G_n. In our construction, the first row corresponds to the ground state chain with all spin set to 0. Given a $m \times n$ matrix A, let $B = (A\ \sigma)$ be the following $m \times (n+1)$ matrix: $B_{i,j} = A_{i,j}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $B_{i,j} = \sigma$, otherwise. Then, we construct the matrix G_n recursively in a ordered way as follows: let

$$G_1 = \begin{pmatrix} 0 \\ + \\ - \end{pmatrix}, \quad G_2 = \begin{pmatrix} 00 \\ +0 \\ -0 \\ 0+ \\ 0- \end{pmatrix} \quad \text{and} \quad G_n = \begin{pmatrix} G_{n,0} \\ G_{n,+} \\ G_{n,-} \end{pmatrix}, \quad n = 3, 4, \ldots,$$

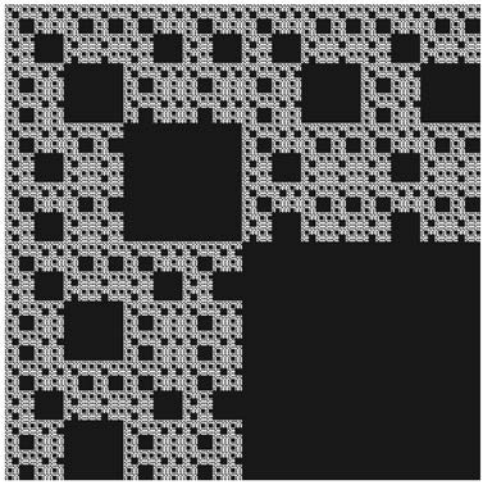

Fig. 1 The digital image associated to the F_9 at the A–D phase boundary

where $G_{n,0} = (G_{n-1}\ 0)$ and $G_{n,\pm} = (G_{n-1,0}\ \pm)$. Notice that $|G_{n,0}| = |G_{n-1}| = N_1(n-1)$ and $|G_{n,\pm}| = |G_{n-1,0}| = |G_{n-2}| = N_1(n-2)$. Furthermore, if we enumerate the $N_1(n)$ rows of G_n from top to bottom then $F_n(i,j)$ is the compatibility condition between the ground states i and j. In particular, $i = 1$ corresponds to the ground state where all spins are zero, which implies $F_n(1,k) = F_n(k,1) = 1$, for all k. Therefore, F_n will have the following block form:

$$F_n = \begin{pmatrix} F_{n-1} & H_{n-1} \\ H_{n-1}^t & 0 \end{pmatrix},$$

where H_{n-1} is a $N_1(n-1) \times (N_1(n) - N_1(n-1))$ matrix and H_{n-1}^t is its transpose. This follows from the following fact: the first block of the first line of F_n gives the compatibility among the elements of $G_{n,0}$, which is the same as the one among the elements of G_{n-1}. On the other hand, the second block in the second line of F_n gives the compatibility among the elements of $G_{n,\pm}$, which are all incompatible. Since the compatibility condition is symmetric, the second block of the first line of F_n and the first block of the second line must be transpose of each other. They give the compatibility between elements of $G_{n,0}$ and $G_{n,\pm}$.

It is easy to see that the matrices F_{n-2}, F_{n-3} and H_{n-3} appear as blocks of the H_{n-1} matrix showing a self-similar structure in F_n, which is illustrated in Fig. 1, where 0's are represented by black and 1's by white pixels.

For the F–A–D phase boundary, $G_{n,0} = (G_{n-1,0}\ 0)$ and $G_{n,\pm} = \left(\begin{smallmatrix} (G_{n-1,0}\ \pm) \\ (G_{n-1,\pm}\ \pm) \end{smallmatrix}\right)$.

5 Numerical Computation of S on the A–D and F–A–D Phase Boundaries

Based on Theorem 3.1, for each strip of width n we define

$$\overline{S}_n = \frac{1}{2}\left(\frac{\ln \lambda_{1,n}}{n+1} + \frac{\ln \lambda_{1,n}}{n}\right),$$

which is an approximation for S, within an error of at most $(\ln \lambda_{1,n})/(2n(n+1)) \equiv \epsilon_n$. The Tables 1 and 2 below give the numerical values of $\overline{S}_n$ for the A–D and F–A–D phase boundaries, respectively.

Table 1 Numerical calculations for the A–D phase boundary

n	$\lambda_{1,n}$	$\frac{\ln \lambda_{1,n}}{n+1}$	$\frac{\ln \lambda_{1,n}}{n}$	$\overline{S}_n$	ϵ_n
4	11.4833	0.488179	0.610223	0.549201	0.061022
5	20.614	0.504328	0.605194	0.554761	0.0505533
6	37.0048	0.515864	0.601841	0.558853	0.0429850
7	66.4285	0.524516	0.599447	0.561981	0.0374655
8	119.248	0.531245	0.597651	0.561981	0.0332030
9	214.066	0.536628	0.596254	0.566626	0.0298130

Table 2 Numerical calculations for the F–A–D phase boundary

n	$\lambda_{1,n}$	$\frac{\ln \lambda_{1,n}}{n+1}$	$\frac{\ln \lambda_{1,n}}{n}$	$\overline{S}_n$	ϵ_n
4	23.92510	0.634985	0.793732	0.714358	0.0793373
5	51.5229	0.657004	0.788405	0.722705	0.0657004
6	110.9550	0.672732	0.784854	0.728793	0.0560610
7	238.942	0.684528	0.782317	0.733423	0.0488948
8	514.564	0.693702	0.780415	0.737059	0.0433564
9	1108.12	0.701042	0.778935	0.739988	0.0389468

To compute the largest eigenvalue of F_n we use the matrix representation given in Sect. 4. Observe that F_n is a matrix of order $N_1(n)$, which grows exponentially fast with n, see Lemma 4.1. This makes harder to compute $\lambda_{1,n}$ as n grows. By taking the values of $\overline{S}_9$ given in the Tables 1 and 2, we have that the two dimensional residual entropy of the BEG model at the A–D and F–A–D phase boundaries are $S_{A-D} = 0.567 \pm 0.030$ and $S_{F-A-D} = 0.740 \pm 0.039$, respectively.

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